MMAT5360 200302

Minimax theorem: $V_r(A) = V_c(A)$

Thm: Either 1. $\exists\, \vec{x} \in P^m$ s.t. $\vec{x}A > \vec{0}$ $(V_r(A) > 0)$, or

2. $\exists\, \vec{y} \in P^n$ s.t. $A\vec{y}^T \leq \vec{0}$ $(V_c(A) \leq 0)$.

Easy to prove $V_r(A) \leq V_c(A)$

Proof of Minimax theorem:

Suppose $\underline{V_r(A) < k < V_c(A)}$ for some $k \in \mathbb{R}$

Let A' be the matrix obtained by subtracting k from every entry of A.

$$V_r(A') = V_r(A) - k < 0 < V_c(A) - k = V_c(A')$$

Contradiction.

Therefore we proved that $v_r(A) = v_c(A)$ $\square$

Bimatrix Games

Normal form

$$(A,B) = \begin{pmatrix} (a_{11}, b_{11}), & (a_{12}, b_{12}) & \cdots & (a_{1n}, b_{1n}) \\ \vdots & & & \\ \vdots & & & \\ (a_{m1}, b_{m1}) & \cdots & \cdots & (a_{mn}, b_{mn}) \end{pmatrix}$$

A, B : $m \times n$ matrices

Payoff matrix of the player

Expected payoffs

Player I : $\pi(\vec{x}, \vec{y}) = \vec{x} A \vec{y}^T$ $\vec{x} \in P^m, \vec{y} \in P^n$

Player II : $\rho(\vec{x}, \vec{y}) = \vec{x} B \vec{y}^T$

Security Level (Safety level)

$$\mu = \max_{\vec{x} \in \mathcal{P}^m} \left(\min_{\vec{y} \in \mathcal{P}^n} \vec{x} A \vec{y}^T \right) = v(A)$$

$$\nu = \max_{\vec{y} \in \mathcal{P}^n} \left(\min_{\vec{x} \in \mathcal{P}^m} \vec{x} B \vec{y}^T \right) = v(B^T)$$

$$\vec{x} B \vec{y}^T = \vec{y} B^T \vec{x}^T$$

Nash equilibrium:

$(\vec{p}, \vec{q}) \in \mathcal{P}^m \times \mathcal{P}^n$ s.t.

$$\vec{x} A \vec{q}^T \le \vec{p} A \vec{q}^T \quad \forall \vec{x} \in \mathcal{P}^m$$

and $$\vec{p} B \vec{y}^T \le \vec{p} B \vec{q}^T \quad \forall \vec{y} \in \mathcal{P}^n$$

2×2 Bimatrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Let $\vec{x} = (x, 1-x)$, $\vec{y} = (y, 1-y)$, $0 \le x, y \le 1$

Let $\vec{x}=(x,1-x)$, $\vec{y}=(y,1-y)$, $0\le x,y\le 1$

Let $P=\{(x,y):\pi(x,y)$ attains its maximum at x for fixed $y.\}$

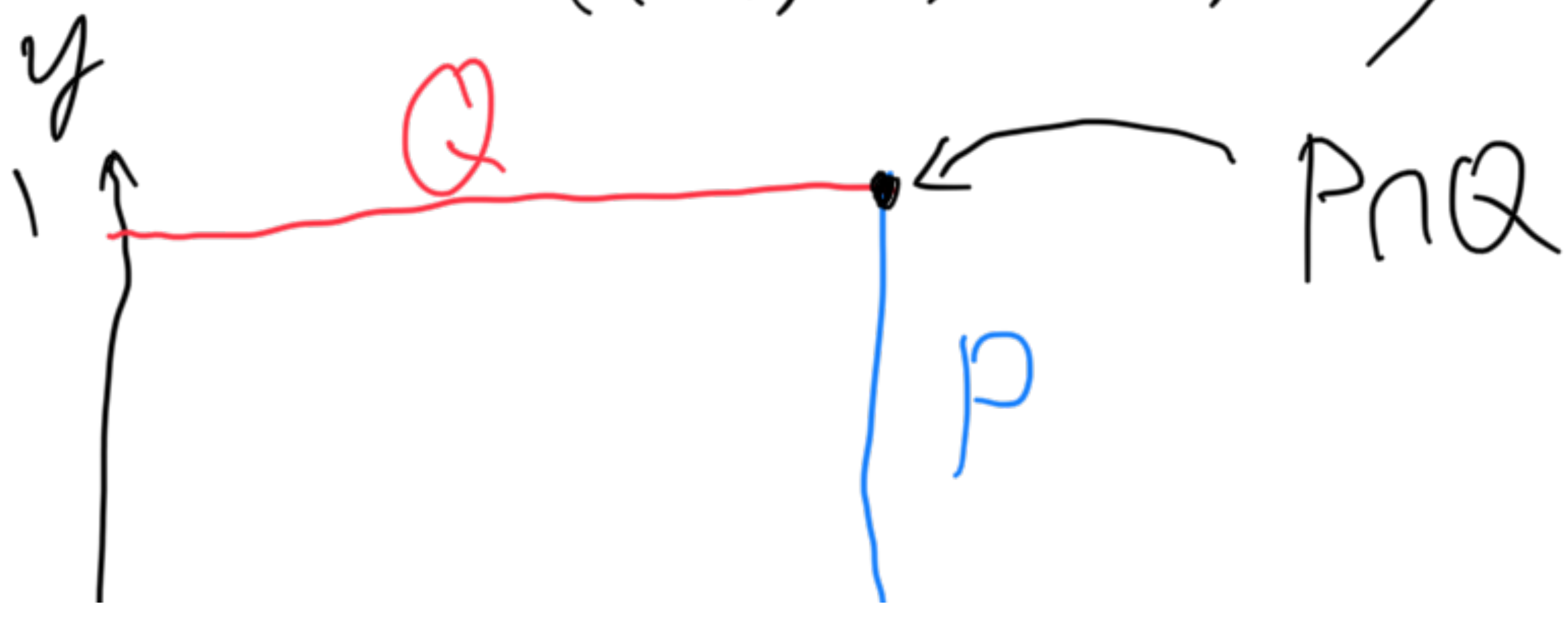
$Q=\{(x,y):\rho(x,y)$ attains its maximum at y for fixed $x.\}$

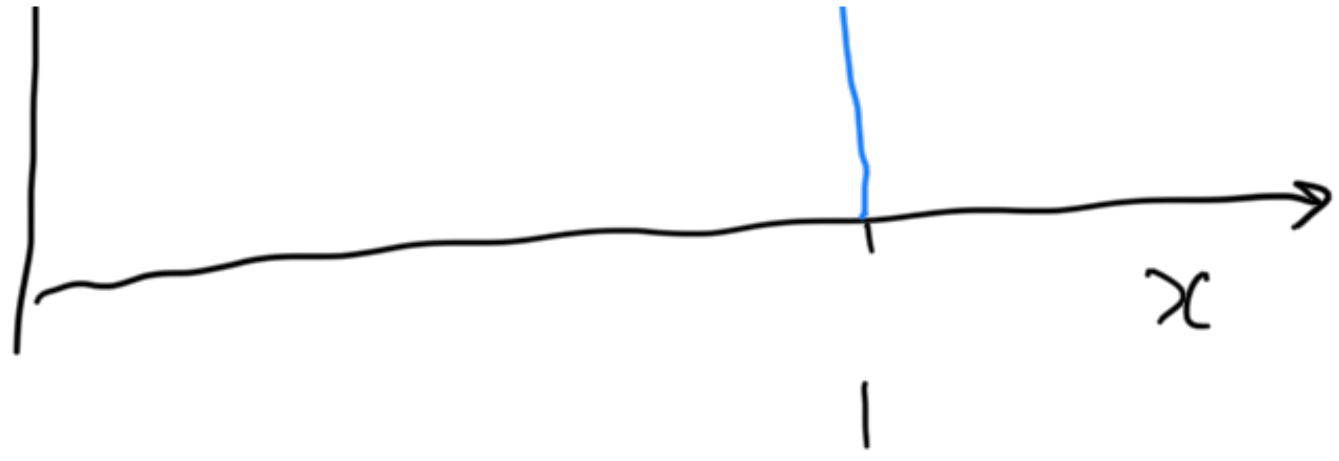
$(\vec{x},\vec{y})$ is a Nash equilibrium $\iff$ $(x,y)\in P\cap Q$

Example

1. Prisoner dilemma

$$(A,B)=\begin{pmatrix}(-5,-5) & (-1,-10)\\ (-10,-1) & (-2,-2)\end{pmatrix}$$





$$\pi(x,y) = (x, 1-x)\begin{pmatrix} -5 & -1 \\ -10 & -2 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = (x, 1-x)\begin{pmatrix} -4y-1 \\ -8y-2 \end{pmatrix}$$

$$P = \{(1,y) : 0 \le y \le 1\}$$

$$\rho(x,y) = (x, 1-x)\begin{pmatrix} -5 & -10 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = (-4x-1, -8x-2)\begin{pmatrix} y \\ 1-y \end{pmatrix}$$

$$Q = \{(x,1) : 0 \le x \le 1\}$$

There is one Nash equilibrium $((1,0), (1,0))$

2. Dating game

$$(A,B) = \begin{pmatrix} (4,2) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}$$

Nash equilibrium

$((1,0), (1,0))$

$((0,1), (0,1))$

$$\pi(x,y) = (x, 1-x)\begin{pmatrix} 4 & 0 \\ \end{pmatrix}\begin{pmatrix} y \\ \end{pmatrix} = (x, 1-x)\begin{pmatrix} 4y \\ \end{pmatrix}$$

$$\cdots \begin{pmatrix} \cdots \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cdots \\ 1-y \end{pmatrix} \quad \cdots \begin{pmatrix} \cdots \\ 1-y \end{pmatrix}$$

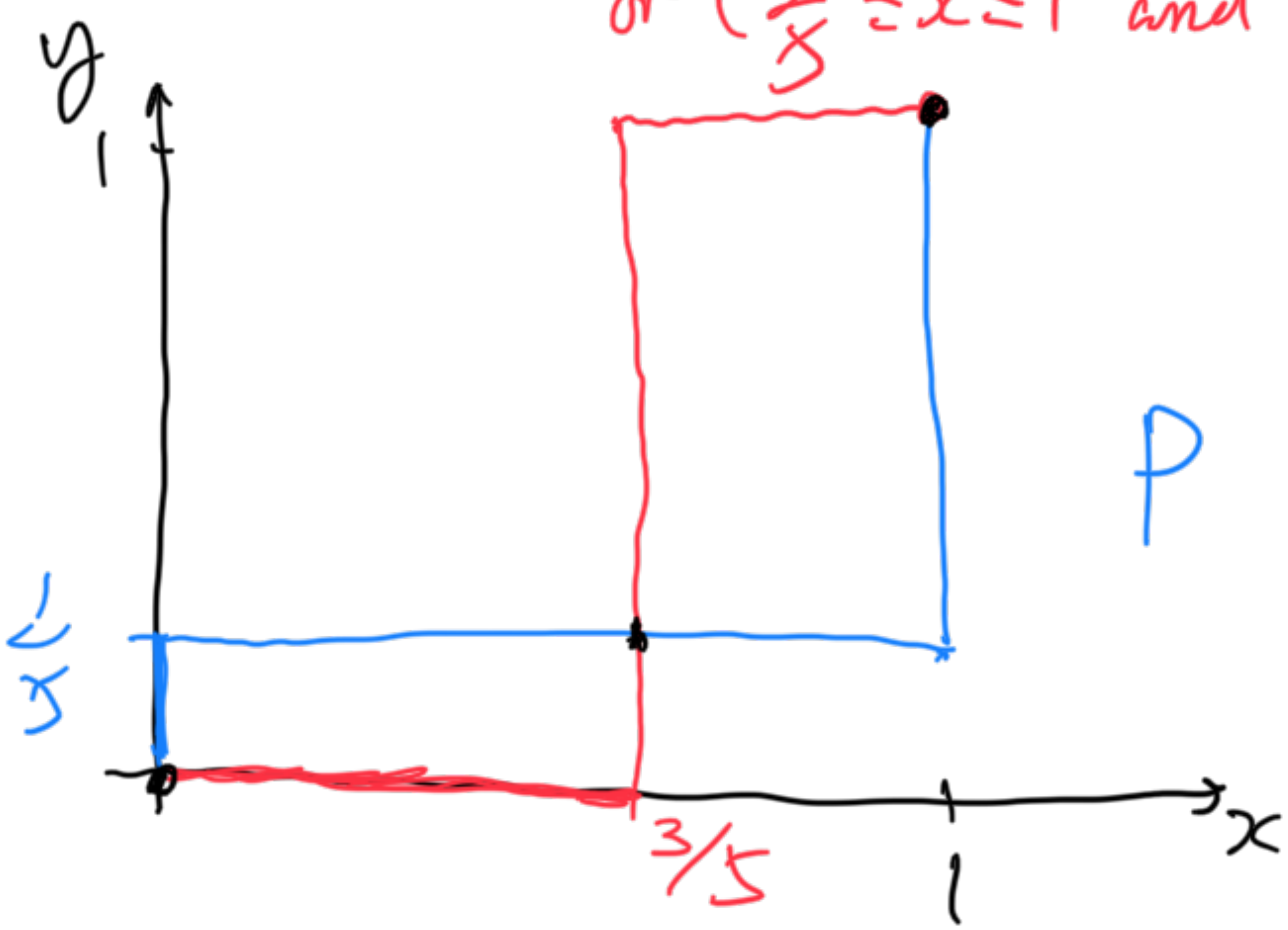
$$P = \{(x,y) : (x=0 \text{ and } 0 \le y \le \tfrac{1}{5}) \text{ or } (0 \le x \le 1 \text{ and } y = \tfrac{1}{5}) \text{ or } (x = 1 \text{ and } \tfrac{1}{5} \le y \le 1\}$$

$$4y = 1-y$$
$$y = \tfrac{1}{5}$$

$$\rho(x,y) = (x, 1-x)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = (2x, 3-3x)\begin{pmatrix} y \\ 1-y \end{pmatrix}$$

$$Q = \{(x,y) : (0 \le x \le \tfrac{3}{5} \text{ and } y=0) \text{ or } (x = \tfrac{3}{5} \text{ and } 0 \le y \le 1) \text{ or } (\tfrac{3}{5} \le x \le 1 \text{ and } y = 1)$$

$$2x = 3-3x$$
$$x = \tfrac{3}{5}$$



3 Nash equilibria

$((1,0),(1,0))$

$((0,1),(0,1))$

$((\tfrac{3}{5},\tfrac{2}{5}),(\tfrac{1}{5},\tfrac{4}{5}))$

$$\vec{p}B = (\tfrac{3}{5}, \tfrac{2}{5})\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = (\tfrac{6}{5}, \tfrac{6}{5})$$

$$A\vec{q}^{\,T} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 4/5 \end{pmatrix}$$

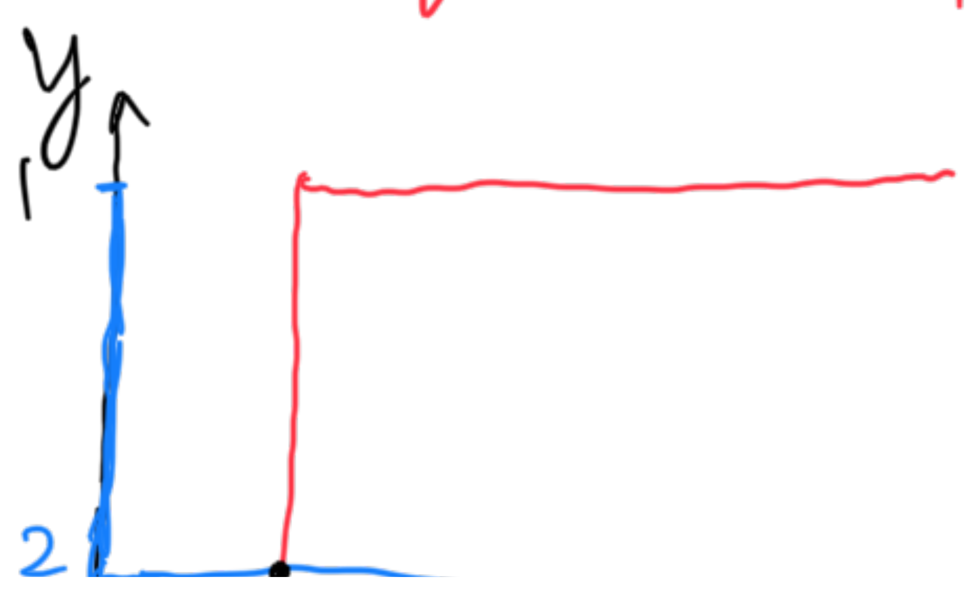
3. $$(A,B) = \begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$$

$$\pi(x,y) = (x, 1-x)\begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = (x, 1-x)\begin{pmatrix} -4y+5 \\ y+3 \end{pmatrix}$$

$$P = \{(x,y) : (x=1 \text{ and } 0 \le y \le \tfrac{2}{5}) \text{ or } (0 \le x \le 1 \text{ and } y = \tfrac{2}{5}) \text{ or } (x=0 \text{ and } \tfrac{2}{5} \le y \le 1)\}$$

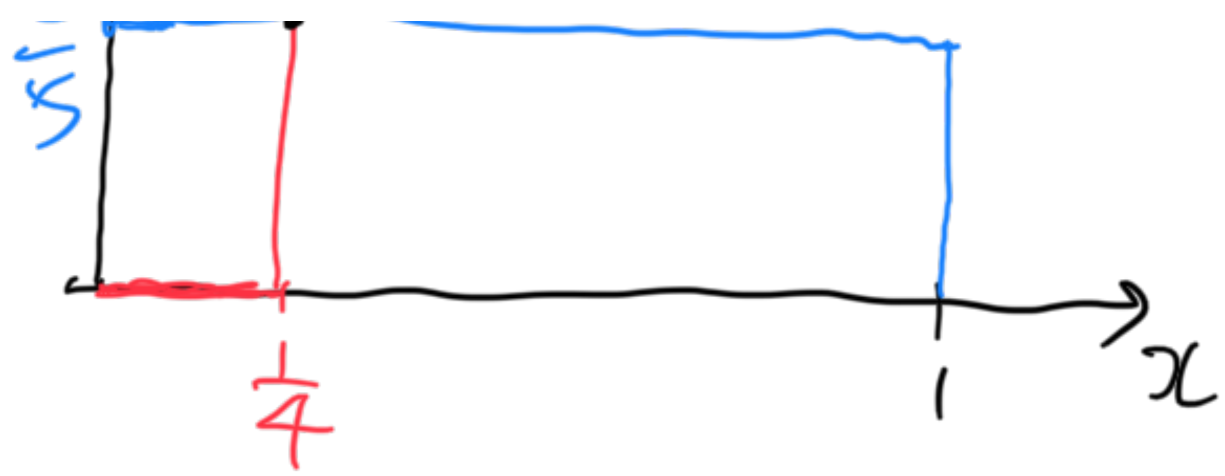
$$\rho(x,y) = (x, 1-x)\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = (2x+2, -2x+3)\begin{pmatrix} y \\ 1-y \end{pmatrix}$$

$$Q = \{(x,y) : (0 \le x \le \tfrac{1}{4} \text{ and } y=0) \text{ or } (x = \tfrac{1}{4} \text{ and } 0 \le y \le 1) \text{ or } (\tfrac{1}{4} \le x \le 1 \text{ and } y=1)\}$$



There is one Nash equilibrium

$$\left((\tfrac{1}{4}, \tfrac{3}{4}), (\tfrac{2}{5}, \tfrac{3}{5}) \right)$$



$v(B^T)$

$$\vec{p}B = \left(\tfrac{1}{4}, \tfrac{3}{4}\right)\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = \left(\tfrac{5}{2}, \tfrac{5}{2}\right)$$

$\vec{p}$ is the minimax strategy of B^T

$$A\vec{q}^{\,T} = \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 17/5 \\ 17/5 \end{pmatrix}$$

$\vec{q}$ is minimax strategy of A

$v(A)$

Nash theorem: Every finite game with finite number of players has at least one Nash equilibrium.

Brouwer fixed point theorem

X: topological space which is homeomorphic

X: topological space which is homeomorphic to the n-dimensional closed unit ball

$B^n = \{ \vec{x} \in \mathbb{R}^n : |\vec{x}| = 1 \}$ $\quad |\vec{x}| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

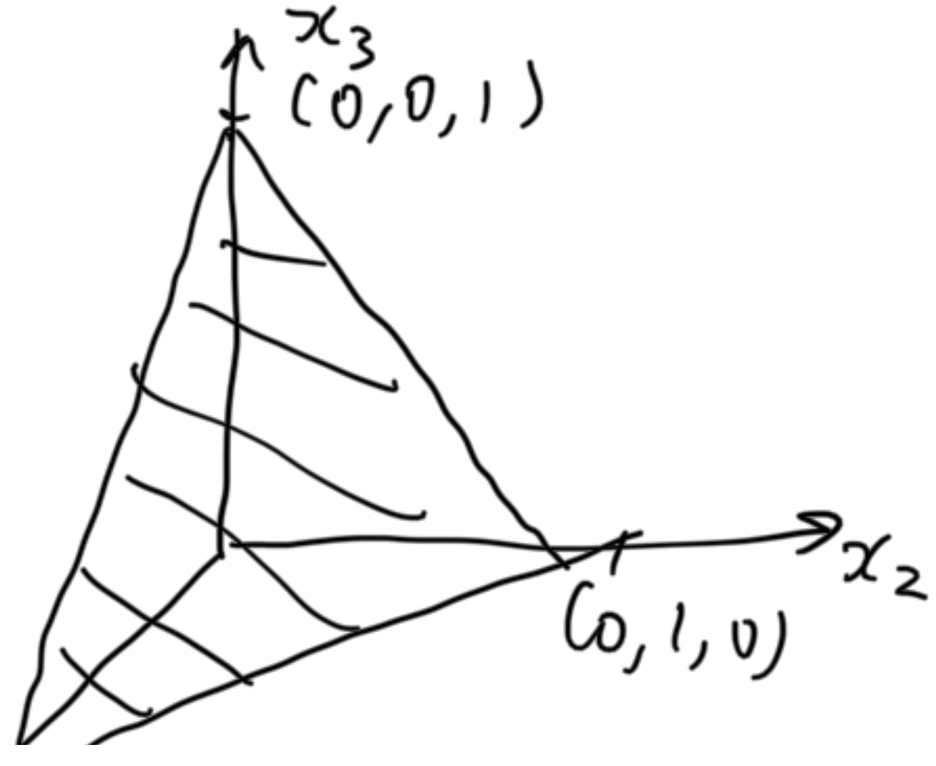
$f: X \to X$ be a continuous map from X into itself.

function

Then f has at least one fixed point, i.e., $\exists\, x \in X$ s.t. $f(x) = x$.

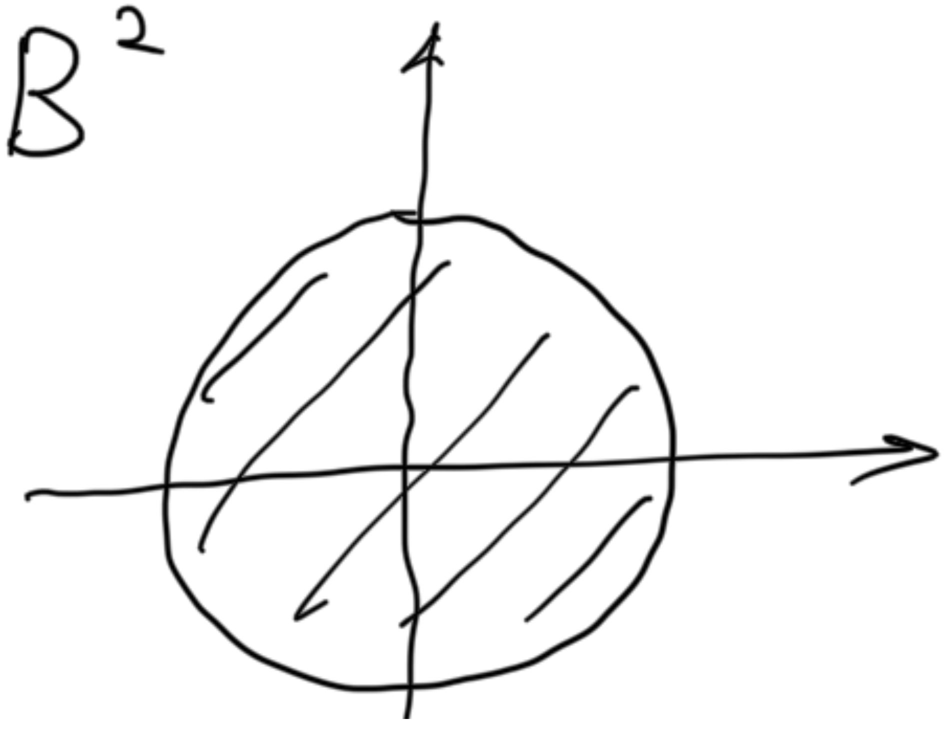
Remark: 1. Δ^n is homeomorphic to B^{n-1}

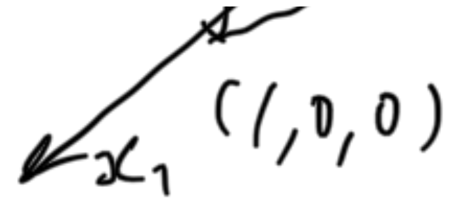
Δ^3



$\cong$

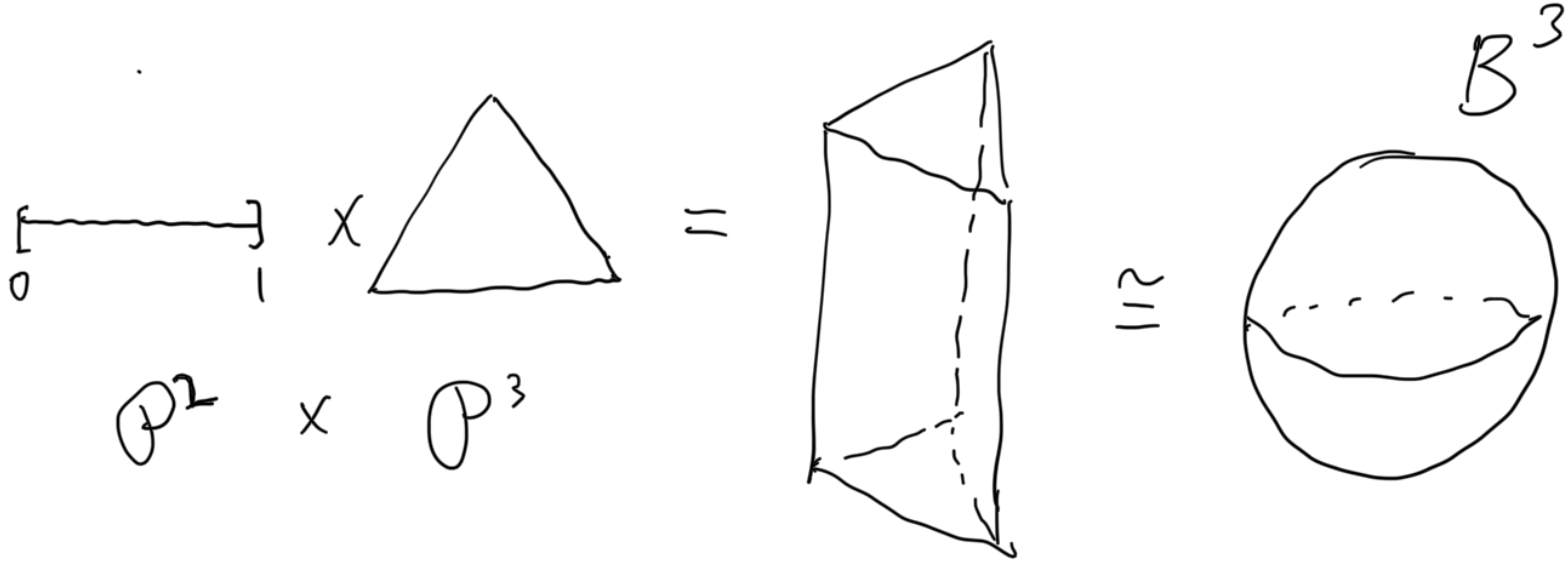
B^2





homeomorphic

$\rho^m \times \rho^n$ is homeomorphic to B^{m+n-2}

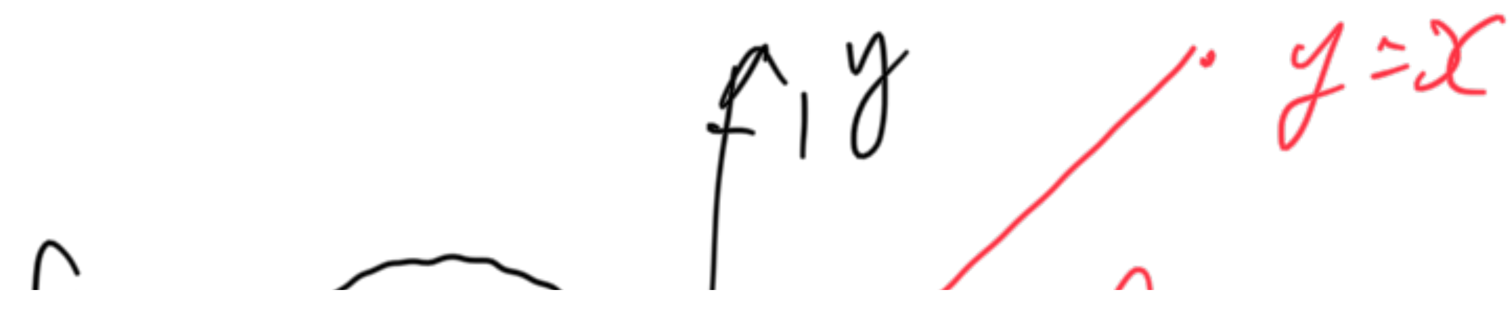


$n=1$, $B^1 = [-1,1]$

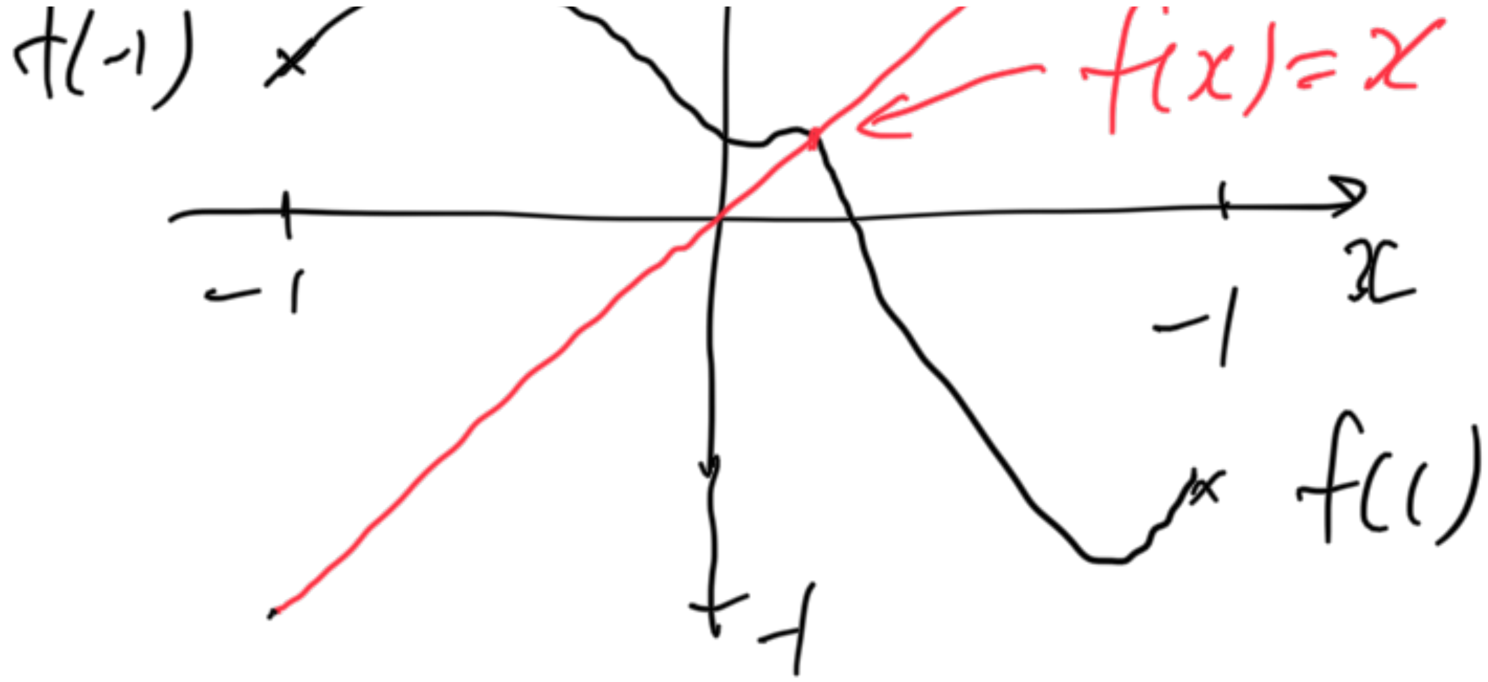
$f : [-1,1] \to [-1,1]$ continuous

Suppose $f(-1) \neq -1$ and $f(1) \neq 1$

Intermediate



value theorem



Let $g(x)=f(x)-x$

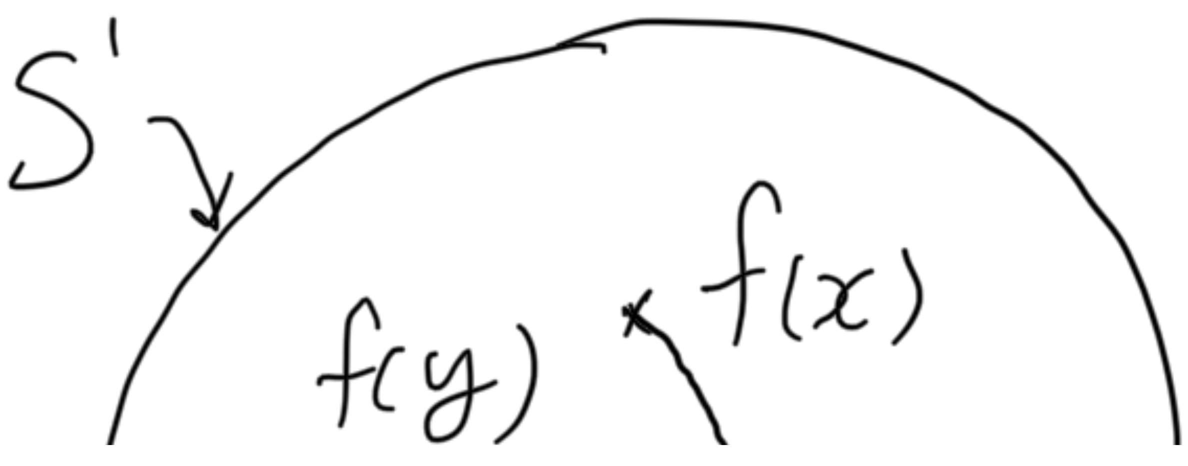
$g(-1)=f(-1)-(-1)>0$

$g(1)=f(1)-1<0$

By IVT, $\exists\ x\in[-1,1]$ s.t. $g(x)=0$

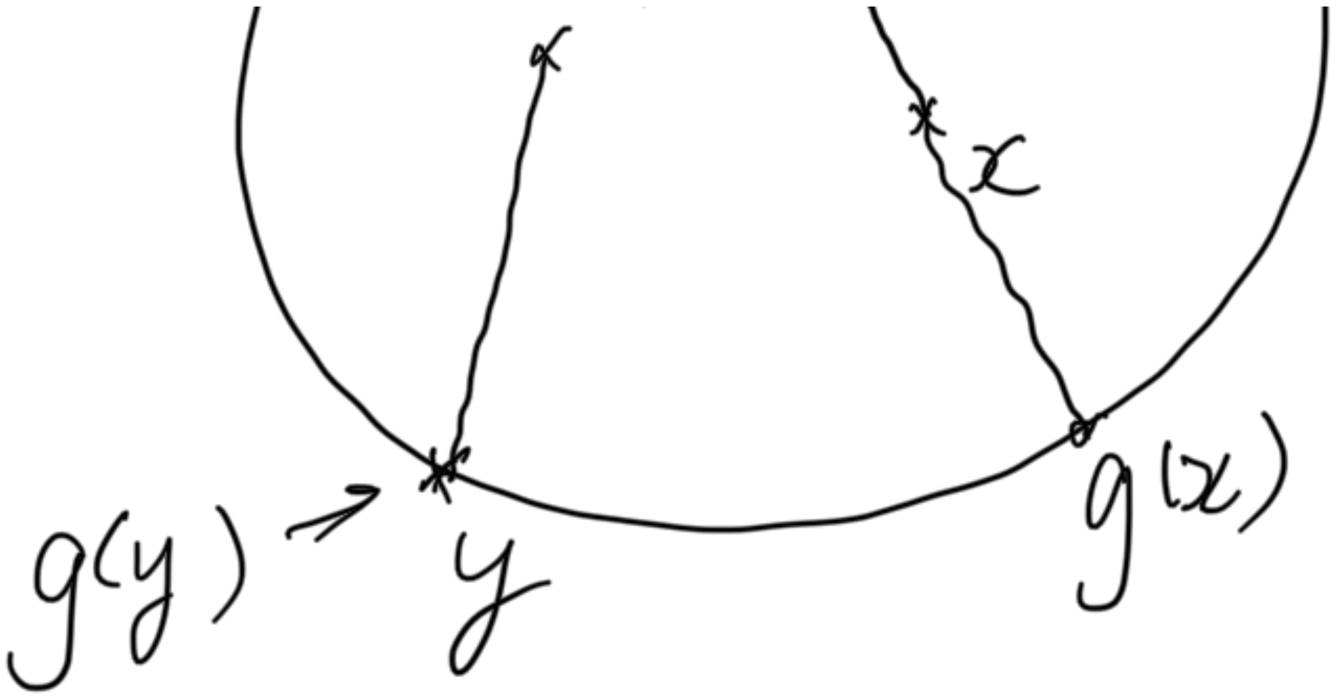
$\therefore\ f(x)=x$ for some $x\in[-1,1]$

$n=2$ $\quad f: B^2 \longrightarrow B^2$ continuous



Suppose $f(x)\neq x \ \forall x\in B^2$

Define $g: B^2\longrightarrow S^1$



$g(x) = x \ \forall x \in S^1$

It is not possible.

Proof of Nash theorem.

For simplicity, we consider 2-person game.

Easy to generalize to general case.

(A, B) : $m \times n$ bimatrices

Define $f : \mathcal{P}^m \times \mathcal{P}^n \longrightarrow \mathcal{P}^m \times \mathcal{P}^n$

$$f(\vec{x}, \vec{y}) = (\vec{p}, \vec{q})$$

$p_i = x_i + c_i \quad q_j = y_j + d_j \quad \vec{p} \in \mathcal{P}^m, \vec{q} \in \mathcal{P}^n$

$$p_i = \frac{\cdots}{1+\sum_{k=1}^{m} c_k}, \quad q_j = \frac{\cdots}{1+\sum_{l=1}^{n} d_l} \qquad p \in \mathcal{P}, \ q \in \mathcal{P}$$

$$c_i = \max\{\hat{e}_i A \vec{y}^T - \vec{x} A \vec{y}^T, 0\}$$

$$d_j = \max\{\vec{x} B \hat{e}_j^T - \vec{x} B \vec{y}^T, 0\}$$

f is continuous.

By Brouwer's fixed point theorem,

$\exists (\vec{p}, \vec{q}) \in \mathcal{P}^m \times \mathcal{P}^n$ s.t. $f(\vec{p}, \vec{q}) = (\vec{p}, \vec{q})$

It remains to prove that $(\vec{p}, \vec{q})$ is a Nash equilibrium.

Suppose not.

WLOG $\exists \vec{x} \in \mathcal{P}^m$ s.t. $\vec{x} A \vec{q}^T > \vec{p} A \vec{q}^T$

$$\vec{p}A\vec{q}^{T} < \vec{x}A\vec{q}^{T} = \sum x_i \hat{e}_i A\vec{q}^{T} \qquad \vec{x} = x_1\hat{e}_1 + \cdots + x_m\hat{e}_m$$

$$x_1 + x_2 + \cdots + x_m = 1$$

$\Rightarrow \exists \quad k = 1, 2, \cdots, m$ s.t.

$$\hat{e}_k A\vec{q}^{T} > \vec{p}A\vec{q}^{T}$$

$$\Rightarrow \quad c_k > 0 \quad \Rightarrow \quad \sum_{i=1}^{m} c_i > 0$$

On the other hand,

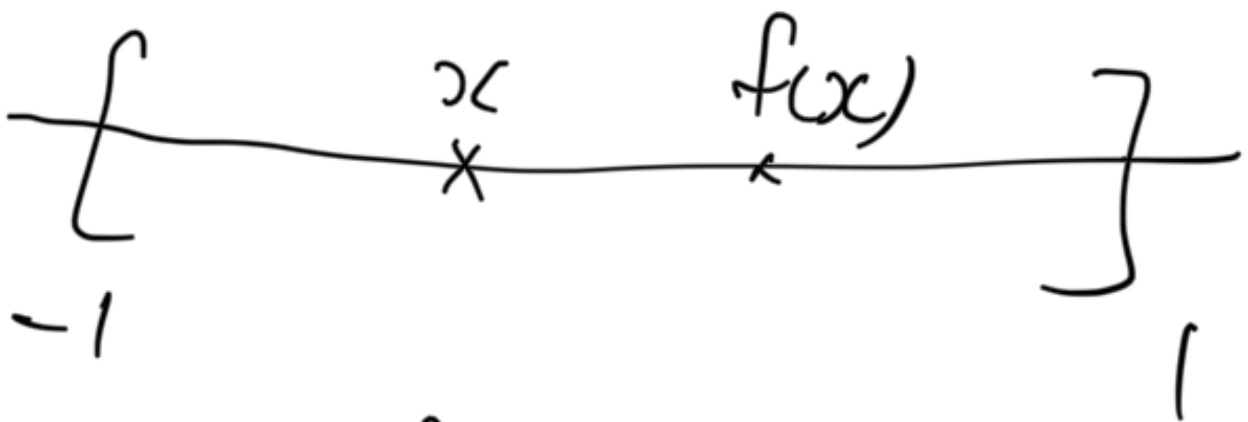
$$\vec{p}A\vec{q}^{T} = \sum p_i \hat{e}_i A\vec{q}^{T}$$

$\exists \quad i = 1, 2, \cdots, m$ s.t. $\hat{e}_i A\vec{q}^{T} \le \vec{p}A\vec{q}^{T} \Rightarrow c_i = 0$

$$p_i = \frac{x_i + c_i}{1 + \sum c} = \frac{x_i}{1 + \sum c} < x_i$$

Contradict that $\vec{p}$ is a fixed point. $\square$

$n=1$ $\qquad B^1=[-1,1]$ $\qquad S^0=\{-1,1\}$



Suppose $f(x) \neq x \quad \forall x \in [-1,1]$

$g : [-1,1] \to \{-1,1\}$ continuous

$$g(x) = \begin{cases} -1 & \text{if } x < f(x) \\ 1 & \text{if } x > f(x) \end{cases}$$

$g(-1) = -1$ and $g(1) = 1$